

Problem Set: The Romer Endogenous Growth Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Time is discrete. There is no population growth:

$$\bar{L}_t = \bar{L}, \quad g_L = 0.$$

The economy has two labour uses: final-goods production and R&D,

$$L_{y,t} + L_{a,t} = \bar{L}, \quad L_{a,t} = \bar{\ell}\bar{L}, \quad L_{y,t} = (1 - \bar{\ell})\bar{L}, \quad 0 < \bar{\ell} < 1.$$

The final-good sector and idea-production sector are

$$Y_t = A_t K_t^\alpha L_{y,t}^{1-\alpha}, \quad 0 < \alpha < 1,$$

$$A_{t+1} - A_t = \bar{z} A_t L_{a,t}, \quad \bar{z} > 0.$$

Capital accumulation is

$$K_{t+1} - K_t = sY_t - \delta K_t, \quad 0 < s < 1, \quad 0 < \delta < 1.$$

Question 1: Nonrival Ideas, Increasing Returns, and Balanced Growth [Total: 60 marks]

- (a) Holding A_t fixed, prove that final-good production has constant returns to scale in rival inputs $(K_t, L_{y,t})$. Then show that if ideas are scaled together with rival inputs, production displays increasing returns:

$$F(2K_t, 2L_{y,t}, 2A_t) > 2F(K_t, L_{y,t}, A_t).$$

Explain the role of nonrivalry of ideas.

- (b) Derive output per person,

$$y_t \equiv \frac{Y_t}{\bar{L}},$$

as a function of A_t , K_t/\bar{L} , and $\bar{\ell}$. Show that a higher stock of ideas raises every person's output because ideas are nonrival.

- (c) Derive the growth rate of ideas,

$$g_A \equiv \frac{A_{t+1} - A_t}{A_t}.$$

Then derive g_Y from the production function using growth-rate algebra. Explain why $g_{L_y} = 0$.

- (d) Use capital accumulation to show

$$g_K = s \frac{Y_t}{K_t} - \delta.$$

Along a balanced-growth path, Y_t/K_t is constant. Use this to prove that $g_K = g_Y$ and derive

$$g_Y = g_K = g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1 - \alpha}.$$

- (e) Compute g_A , g_Y , and g_y when

$$\alpha = \frac{1}{3}, \quad \bar{z} = 0.0005, \quad \bar{\ell} = 0.04, \quad \bar{L} = 1000.$$

Then interpret the "scale effect" in this model.

Question 2: Steady-State Ratios, Labour Allocation, and Policy Trade-offs [Total: 40 marks]

- (a) Use the balanced-growth condition and capital accumulation to derive the steady-state output-capital ratio:

$$\frac{Y^*}{K^*} = \frac{g_K + \delta}{s}.$$

Then derive steady-state capital per person:

$$k_t^* \equiv \frac{K_t^*}{\bar{L}} = \left(\frac{s}{g_K + \delta} \right)^{1/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}).$$

- (b) Derive steady-state output per person:

$$y_t^* = \left(\frac{s}{g_K + \delta} \right)^{\alpha/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}).$$

Explain why y_t^* grows even though population is constant.

- (c) Derive the effect of $\bar{\ell}$ on the long-run growth rate g_y . Then discuss the static cost of increasing $\bar{\ell}$ through the term $(1 - \bar{\ell})$ in y_t^* . Interpret the intertemporal trade-off.
- (d) Suppose $\alpha = 1/3$, $s = 0.24$, $\delta = 0.05$, $\bar{z} = 0.0005$, $\bar{L} = 1000$, and $\bar{\ell} = 0.04$. Compute g_K and Y^*/K^* . Then derive the general comparative statics of g_y with respect to \bar{z} , $\bar{\ell}$, \bar{L} , and α .

Detailed Solutions

Solution to Question 1

[60 marks]

Part (a)

[12 marks]

The final-good production function is

$$Y_t = F(K_t, L_{y,t}, A_t) = A_t K_t^\alpha L_{y,t}^{1-\alpha}, \quad 0 < \alpha < 1.$$

First hold A_t fixed and scale only rival inputs $(K_t, L_{y,t})$ by a positive scalar $\lambda > 0$:

$$F(\lambda K_t, \lambda L_{y,t}, A_t) = A_t (\lambda K_t)^\alpha (\lambda L_{y,t})^{1-\alpha}.$$

Using the laws of exponents:

$$F(\lambda K_t, \lambda L_{y,t}, A_t) = A_t \lambda^\alpha K_t^\alpha \lambda^{1-\alpha} L_{y,t}^{1-\alpha} = \lambda^{\alpha+1-\alpha} A_t K_t^\alpha L_{y,t}^{1-\alpha}.$$

Hence,

$$\boxed{F(\lambda K_t, \lambda L_{y,t}, A_t) = \lambda F(K_t, L_{y,t}, A_t)}.$$

Therefore, holding knowledge fixed, the final-good technology has constant returns to scale in the rival inputs capital and labour.

Now scale ideas together with rival inputs. For $\lambda > 1$:

$$F(\lambda K_t, \lambda L_{y,t}, \lambda A_t) = \lambda A_t (\lambda K_t)^\alpha (\lambda L_{y,t})^{1-\alpha}.$$

Thus,

$$F(\lambda K_t, \lambda L_{y,t}, \lambda A_t) = \lambda A_t \lambda^\alpha K_t^\alpha \lambda^{1-\alpha} L_{y,t}^{1-\alpha} = \lambda^2 F(K_t, L_{y,t}, A_t).$$

For $\lambda = 2$,

$$F(2K_t, 2L_{y,t}, 2A_t) = 4F(K_t, L_{y,t}, A_t).$$

Since

$$4F(K_t, L_{y,t}, A_t) > 2F(K_t, L_{y,t}, A_t),$$

we have

$$\boxed{F(2K_t, 2L_{y,t}, 2A_t) > 2F(K_t, L_{y,t}, A_t)}.$$

The economic reason is the nonrivalry of ideas. Capital and labour are rival inputs: if one firm uses a machine or worker, another firm cannot use the same machine or worker at the same time. Ideas are different. Once an idea is discovered, it can be used by many producers simultaneously. Therefore, replication of rival inputs gives constant returns, but adding the nonrival stock of knowledge creates increasing returns in rival inputs plus ideas.

Marking guide: CRS proof holding A_t fixed, 4; IRS proof scaling A_t , K_t , and $L_{y,t}$, 4; correct inequality for doubling, 2; economic interpretation of nonrivalry, 2.

Part (b)

[10 marks]

Output per person is

$$y_t \equiv \frac{Y_t}{\bar{L}}.$$

Using

$$Y_t = A_t K_t^\alpha L_{y,t}^{1-\alpha},$$

divide both sides by \bar{L} :

$$\frac{Y_t}{\bar{L}} = A_t K_t^\alpha L_{y,t}^{1-\alpha} \bar{L}^{-1}.$$

Write capital as

$$K_t^\alpha = \left(\frac{K_t}{\bar{L}} \bar{L} \right)^\alpha = \left(\frac{K_t}{\bar{L}} \right)^\alpha \bar{L}^\alpha.$$

Similarly,

$$L_{y,t}^{1-\alpha} = \left(\frac{L_{y,t}}{\bar{L}} \bar{L} \right)^{1-\alpha} = \left(\frac{L_{y,t}}{\bar{L}} \right)^{1-\alpha} \bar{L}^{1-\alpha}.$$

Therefore,

$$y_t = A_t \left(\frac{K_t}{\bar{L}} \right)^\alpha \left(\frac{L_{y,t}}{\bar{L}} \right)^{1-\alpha} \bar{L}^\alpha \bar{L}^{1-\alpha} \bar{L}^{-1}.$$

Since

$$\bar{L}^\alpha \bar{L}^{1-\alpha} \bar{L}^{-1} = 1,$$

we obtain

$$y_t = A_t \left(\frac{K_t}{\bar{L}} \right)^\alpha \left(\frac{L_{y,t}}{\bar{L}} \right)^{1-\alpha}.$$

Because

$$L_{y,t} = (1 - \bar{\ell}) \bar{L},$$

we have

$$\frac{L_{y,t}}{\bar{L}} = 1 - \bar{\ell}.$$

Hence,

$$y_t = A_t \left(\frac{K_t}{\bar{L}} \right)^\alpha (1 - \bar{\ell})^{1-\alpha}.$$

A higher A_t raises output per person one-for-one, holding capital per person and the labour allocation fixed. This is the key implication of nonrivalry: the same idea can be used simultaneously by all producers, so the total stock of ideas, not ideas per person, enters per-capita output.

Marking guide: correct definition of y_t , 2; algebraic derivation, 5; substitution of $L_{y,t}/\bar{L} = 1 - \bar{\ell}$, 1; interpretation of nonrival ideas, 2.

Part (c)

[12 marks]

The law of motion for ideas is

$$A_{t+1} - A_t = \bar{z} A_t L_{a,t}.$$

Divide by A_t :

$$\frac{A_{t+1} - A_t}{A_t} = \bar{z} L_{a,t}.$$

Thus,

$$g_A = \bar{z} L_{a,t}.$$

Because

$$L_{a,t} = \bar{\ell} \bar{L},$$

we obtain

$$g_A = \bar{z} \bar{\ell} \bar{L}.$$

Therefore, the growth rate of ideas is higher when R&D productivity \bar{z} is higher, when a larger fraction of labour is allocated to research, and when the total population is larger.

Now use the production function:

$$Y_t = A_t K_t^\alpha L_{y,t}^{1-\alpha}.$$

Using growth-rate algebra,

$$g_Y = g_A + \alpha g_K + (1 - \alpha) g_{L_y}.$$

Because population is constant and the labour share allocated to final-good production is constant,

$$L_{y,t} = (1 - \bar{\ell}) \bar{L},$$

so

$$g_{L_y} = 0.$$

Hence,

$$g_Y = g_A + \alpha g_K.$$

Substituting $g_A = \bar{z} \bar{\ell} \bar{L}$:

$$g_Y = \bar{z} \bar{\ell} \bar{L} + \alpha g_K.$$

Marking guide: derivation of g_A , 4; substitution of $L_{a,t} = \bar{\ell} \bar{L}$, 2; growth-rate decomposition of Y_t , 4; explanation of $g_{L_y} = 0$, 2.

Part (d)

[18 marks]

Capital accumulation is

$$K_{t+1} - K_t = sY_t - \delta K_t.$$

Divide both sides by K_t :

$$\frac{K_{t+1} - K_t}{K_t} = s \frac{Y_t}{K_t} - \delta.$$

Thus,

$$g_K = s \frac{Y_t}{K_t} - \delta.$$

Along a balanced-growth path, the output-capital ratio is constant:

$$\frac{Y_t}{K_t} = \text{constant}.$$

Taking growth rates:

$$g_{Y/K} = 0.$$

Since

$$g_{Y/K} = g_Y - g_K,$$

we have

$$0 = g_Y - g_K.$$

Therefore,

$$g_Y = g_K.$$

From part (c),

$$g_Y = g_A + \alpha g_K.$$

Using $g_K = g_Y$:

$$g_Y = g_A + \alpha g_Y.$$

Rearrange:

$$(1 - \alpha)g_Y = g_A.$$

Therefore,

$$g_Y = \frac{g_A}{1 - \alpha}.$$

Since

$$g_A = \bar{z}\bar{\ell}\bar{L},$$

we obtain

$$g_Y = g_K = \frac{\bar{z}\bar{\ell}\bar{L}}{1 - \alpha}.$$

Output per person is

$$y_t = \frac{Y_t}{L}.$$

Because \bar{L} is constant,

$$g_y = g_Y - g_L = g_Y.$$

Since $g_L = 0$,

$$g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1 - \alpha}.$$

Therefore,

$$g_Y = g_K = g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1 - \alpha}.$$

The key result is that the economy has sustained per-capita growth even with no population growth. Growth is sustained because the stock of nonrival ideas grows endogenously through R&D, and these ideas raise the productivity of all producers.

Marking guide: derivation of g_K , 4; use of constant Y_t/K_t , 4; proof that $g_Y = g_K$, 3; solution for g_Y and g_K , 5; derivation of g_y , 2.

Part (e)

[8 marks]

The parameter values are

$$\alpha = \frac{1}{3}, \quad \bar{z} = 0.0005, \quad \bar{\ell} = 0.04, \quad \bar{L} = 1000.$$

The growth rate of ideas is

$$g_A = \bar{z}\bar{\ell}\bar{L}.$$

Thus,

$$g_A = 0.0005 \times 0.04 \times 1000 = 0.02.$$

Hence,

$$\boxed{g_A = 0.02.}$$

Since

$$g_Y = g_y = \frac{g_A}{1 - \alpha},$$

and

$$1 - \alpha = \frac{2}{3},$$

we get

$$g_Y = g_y = \frac{0.02}{2/3} = 0.03.$$

Therefore,

$$\boxed{g_Y = g_y = 0.03.}$$

Aggregate output, aggregate capital, and output per person grow at 3% per period.

The scale effect is the implication that

$$\frac{\partial g_y}{\partial \bar{L}} = \frac{\bar{z}\bar{\ell}}{1 - \alpha} > 0.$$

Thus, a larger population raises the long-run growth rate because there are more potential researchers producing ideas. Since ideas are nonrival, the ideas produced by additional researchers raise productivity for the entire economy. This is the central mechanism behind the Romer model, but it is also a controversial prediction because empirical evidence does not always support a simple positive relationship between country size and long-run per-capita growth.

Marking guide: numerical g_A , 2; numerical g_Y and g_y , 3; interpretation of the scale effect, 3.

Solution to Question 2**[40 marks]**

Part (a)

[12 marks]

Capital accumulation is

$$K_{t+1} - K_t = sY_t - \delta K_t.$$

Divide by K_t :

$$\frac{K_{t+1} - K_t}{K_t} = s \frac{Y_t}{K_t} - \delta.$$

Hence,

$$g_K = s \frac{Y_t}{K_t} - \delta.$$

Solving for the output-capital ratio:

$$s \frac{Y_t}{K_t} = g_K + \delta,$$

so

$$\boxed{\frac{Y^*}{K^*} = \frac{g_K + \delta}{s}}.$$

Now derive steady-state capital per person. The production function is

$$Y_t = A_t K_t^\alpha L_{y,t}^{1-\alpha}.$$

Using

$$L_{y,t} = (1 - \bar{\ell}) \bar{L},$$

we have

$$Y_t = A_t K_t^\alpha (1 - \bar{\ell})^{1-\alpha} \bar{L}^{1-\alpha}.$$

Divide by K_t :

$$\frac{Y_t}{K_t} = A_t K_t^{\alpha-1} (1 - \bar{\ell})^{1-\alpha} \bar{L}^{1-\alpha}.$$

Let

$$k_t \equiv \frac{K_t}{\bar{L}}.$$

Then

$$K_t = k_t \bar{L}.$$

Thus,

$$K_t^{\alpha-1} = (k_t \bar{L})^{\alpha-1} = k_t^{\alpha-1} \bar{L}^{\alpha-1}.$$

Substitute this into Y_t/K_t :

$$\frac{Y_t}{K_t} = A_t k_t^{\alpha-1} \bar{L}^{\alpha-1} (1 - \bar{\ell})^{1-\alpha} \bar{L}^{1-\alpha}.$$

Since

$$\bar{L}^{\alpha-1} \bar{L}^{1-\alpha} = 1,$$

we get

$$\frac{Y_t}{K_t} = A_t k_t^{\alpha-1} (1 - \bar{\ell})^{1-\alpha}.$$

At the balanced-growth path:

$$\frac{g_K + \delta}{s} = A_t (k_t^*)^{\alpha-1} (1 - \bar{\ell})^{1-\alpha}.$$

Rearrange:

$$(k_t^*)^{\alpha-1} = \frac{g_K + \delta}{s} A_t^{-1} (1 - \bar{\ell})^{-(1-\alpha)}.$$

Because $\alpha - 1 = -(1 - \alpha)$,

$$(k_t^*)^{1-\alpha} = \frac{s}{g_K + \delta} A_t (1 - \bar{\ell})^{1-\alpha}.$$

Raise both sides to the power $1/(1 - \alpha)$:

$$k_t^* = \left(\frac{s}{g_K + \delta} \right)^{1/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}).$$

Therefore,

$$k_t^* = \left(\frac{s}{g_K + \delta} \right)^{1/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}).$$

This expression is time-indexed because A_t grows over time. Thus, unlike the Solow model with exogenous technology in effective-labour units, capital per person grows endogenously through the growth of ideas.

Marking guide: output-capital ratio, 4; expression for Y_t/K_t , 4; solution for k_t^* , 4.

Part (b)

[10 marks]

Output per person is

$$y_t = \frac{Y_t}{L} = A_t \left(\frac{K_t}{L} \right)^\alpha (1 - \bar{\ell})^{1-\alpha}.$$

At the balanced-growth path,

$$k_t^* = \left(\frac{s}{g_K + \delta} \right)^{1/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}).$$

Thus,

$$y_t^* = A_t (k_t^*)^\alpha (1 - \bar{\ell})^{1-\alpha}.$$

Substitute k_t^* :

$$y_t^* = A_t \left[\left(\frac{s}{g_K + \delta} \right)^{1/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}) \right]^\alpha (1 - \bar{\ell})^{1-\alpha}.$$

Simplify:

$$y_t^* = A_t \left(\frac{s}{g_K + \delta} \right)^{\alpha/(1-\alpha)} A_t^{\alpha/(1-\alpha)} (1 - \bar{\ell})^\alpha (1 - \bar{\ell})^{1-\alpha}.$$

Since

$$(1 - \bar{\ell})^\alpha (1 - \bar{\ell})^{1-\alpha} = 1 - \bar{\ell},$$

and

$$A_t A_t^{\alpha/(1-\alpha)} = A_t^{1+\alpha/(1-\alpha)} = A_t^{1/(1-\alpha)},$$

we obtain

$$y_t^* = \left(\frac{s}{g_K + \delta} \right)^{\alpha/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}).$$

Output per person grows because A_t grows endogenously through R&D:

$$A_{t+1} - A_t = \bar{z} A_t L_{a,t}.$$

Even though population is constant, a fixed research workforce produces new ideas every period. Since ideas are nonrival, the growing total stock of ideas raises the productivity of the entire economy. Therefore, sustained per-capita growth is possible without exogenous technological progress.

Marking guide: substitution into y_t , 4; algebraic simplification, 4; explanation of why y_t^* grows, 2.

Part (c)

[10 marks]

From Question 1,

$$g_y = \frac{\bar{z} \bar{\ell} \bar{L}}{1 - \alpha}.$$

Differentiate with respect to $\bar{\ell}$:

$$\frac{\partial g_y}{\partial \bar{\ell}} = \frac{\bar{z} \bar{L}}{1 - \alpha} > 0.$$

Thus, allocating a larger fraction of labour to R&D raises the long-run growth rate of output per person. However, steady-state output per person is

$$y_t^* = \left(\frac{s}{g_K + \delta} \right)^{\alpha/(1-\alpha)} A_t^{1/(1-\alpha)} (1 - \bar{\ell}).$$

The term

$$(1 - \bar{\ell})$$

captures the static cost of R&D labour allocation. A larger $\bar{\ell}$ means fewer workers are producing current final goods:

$$\frac{\partial(1 - \bar{\ell})}{\partial \bar{\ell}} = -1.$$

Thus, increasing $\bar{\ell}$ reduces current output mechanically by shrinking final-goods employment. The intertemporal trade-off is therefore:

more R&D labour \Rightarrow lower current final-goods production but faster idea growth.

In the short run, a rise in $\bar{\ell}$ lowers output through the reduction in L_y . In the long run, it raises the growth rate because more researchers produce more ideas. Since ideas are nonrival, the future benefit of research is economy-wide.

Marking guide: derivative of g_y with respect to $\bar{\ell}$, 3; identification of static output cost, 3; intertemporal trade-off interpretation, 4.

Part (d)

[8 marks]

The parameters are

$$\alpha = \frac{1}{3}, \quad s = 0.24, \quad \delta = 0.05, \quad \bar{z} = 0.0005, \quad \bar{L} = 1000, \quad \bar{\ell} = 0.04.$$

From Question 1,

$$g_K = \frac{\bar{z}\bar{\ell}\bar{L}}{1 - \alpha}.$$

Substitute the numerical values:

$$g_K = \frac{0.0005 \times 0.04 \times 1000}{2/3} = \frac{0.02}{2/3} = 0.03.$$

Thus,

$$\boxed{g_K = 0.03.}$$

The steady-state output-capital ratio is

$$\frac{Y^*}{K^*} = \frac{g_K + \delta}{s}.$$

Hence,

$$\frac{Y^*}{K^*} = \frac{0.03 + 0.05}{0.24} = \frac{0.08}{0.24} = 0.3333.$$

Therefore,

$$\boxed{\frac{Y^*}{K^*} \approx 0.3333.}$$

Now consider the general growth expression:

$$g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1 - \alpha}.$$

The comparative statics are:

$$\boxed{\frac{\partial g_y}{\partial \bar{z}} = \frac{\bar{\ell}\bar{L}}{1 - \alpha} > 0,}$$

$$\boxed{\frac{\partial g_y}{\partial \bar{\ell}} = \frac{\bar{z}\bar{L}}{1 - \alpha} > 0,}$$

$$\boxed{\frac{\partial g_y}{\partial \bar{L}} = \frac{\bar{z}\bar{\ell}}{1 - \alpha} > 0,}$$

and

$$\boxed{\frac{\partial g_y}{\partial \alpha} = \frac{\bar{z}\bar{\ell}\bar{L}}{(1 - \alpha)^2} > 0.}$$

Therefore, long-run per-capita growth is increasing in research productivity, the research labour share, the scale of the economy, and the capital elasticity of output.

The last result reflects amplification through capital accumulation. A higher α means that idea growth induces stronger capital deepening, so the growth rate of output exceeds idea growth by a larger multiplier:

$$g_y = \frac{g_A}{1 - \alpha}.$$

Marking guide: numerical g_K , 2; numerical Y^*/K^* , 2; comparative statics with respect to \bar{z} , $\bar{\ell}$, and \bar{L} , 2; comparative static with respect to α and interpretation, 2.